NORTH SYDNEY GIRLS HIGH SCHOOL

2016 HSC Assessment Task 1

Mathematics Extension 2

General Instructions

- Reading Time – 2 minutes
- Working Time – 60 minutes
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- In Questions 6−7, show relevant mathematical reasoning and/or calculations

Total Marks – 41

Section I Pages 2–3

5 marks

- Attempt Questions 1−5
- Allow about 8 minutes for this section

Section II Pages 4−6

36 marks

- Attempt Questions 6−7
- Allow about 52 minutes for this section

Student Name:______________________________

Student Number:____________________________

Teacher:

- ○ Mr T
- ○ Ms Everingham
- ○ Ms Narayanan

QUESTION	MARK
1−5	/5
6	/17
7	/19
TOTAL	/41

Section I

5 marks
Attempt Questions 1–5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5.

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1 Which of the following represents the locus $|z+2-3i| = 4$ on the Argand plane?

(A) A circle with centre $(2,-3)$ and radius 2

(B) A circle with centre $(-2,3)$ and radius 2

(C) A circle with centre $(2,-3)$ and radius 4

(D) A circle with centre $(-2,3)$ and radius 4

2 Which of the following is equal to $(1+i)^{12}$?

(A) 64

(B) $64i$

(C) – 64

(D) $-64i$

3 Given $z_1 = 3\,\mathrm{cis}\theta$ and $z_2 = 7\mathrm{cis}\phi$, which of the following could be $|z_1 + z_2|$?

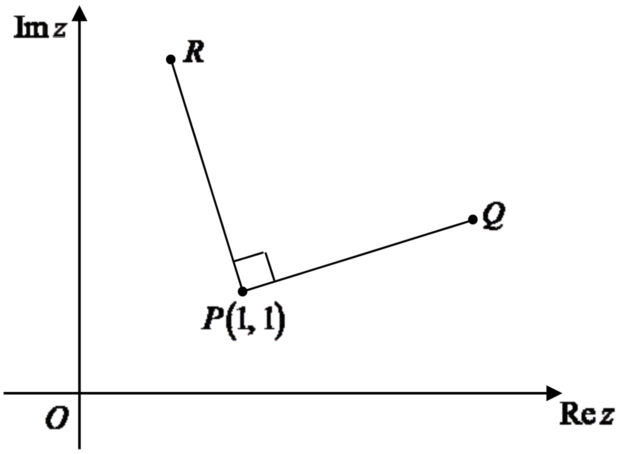
(A) 1

(B) 3

(C) 7

(D) 11

4 Let P be a point on the Argand diagram with coordinates $(1,1)$.
Let Q be another point such that the complex number represented by vector PQ is given by $a+bi$.



The vector PQ is rotated about the point P anti-clockwise by an angle of $\frac{\pi}{2}$ to form the vector PR. What are the coordinates of R?

(A) $(-b, a)$

(B) $(1-b, 1+a)$

(C) $(b, -a)$

(D) $(1+b, 1-a)$

5 The equation $x^2 - 2ix + k = 0$ has a root of $-1+2i$. What is the value of k?

(A) $-1+2i$

(B) $-1-2i$

(C) 1

(D) 5

Section II

36 marks
Attempt Questions 6–7
Allow about 52 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing papers are available.

In Questions 6–7, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (17 marks) Use a SEPARATE writing booklet.

(a) Let $z = 7 - 2i$. Find in $x + iy$ form:

(i) z^2 **1**

(ii) $2i\,\overline{z}$ **1**

(b) (i) Find the square roots of $-5 - 12i$. **3**

(ii) Hence, solve $z^2 + 5iz - 5 + 3i = 0$. **2**

(c) Given that $z = 1 + i$ and $w = 1 - i\sqrt{3}$

(i) Find $\frac{z}{w}$ in the form $x + iy$ **2**

(ii) Find z and w in modulus argument form. **2**

(iii) Use part (ii) to express $\frac{z}{w}$ in modulus argument form. **1**

(iv) Hence, find the exact value of $\sin\frac{7\pi}{12}$. **1**

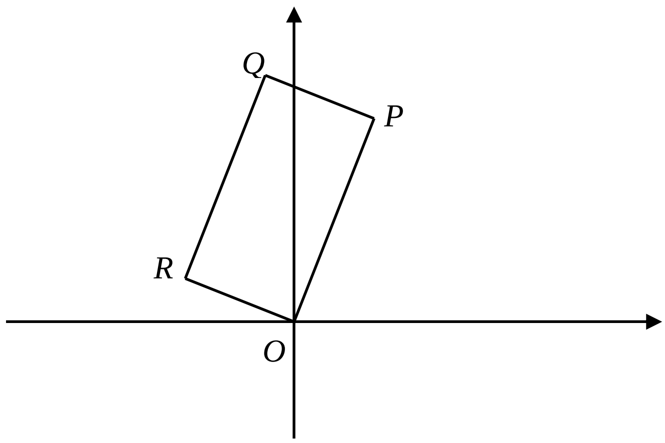
(d) Sketch the region on the Argand plane where **2**

$$|z - 3i| \le 3 \quad \text{and} \quad 0 \le \arg z \le \frac{\pi}{4}$$

Question 6 continues on Page 5

Question 6 (continued)

(e) $OPQR$ is a rectangle and $OP = 2OR$. **2**



If R represents the complex number z and Q represents $-1+5i$, find z.

Question 7 (19 marks) Use a SEPARATE writing booklet.

(a) (i) Find all 6^{th} roots of -1, expressing them in modulus-argument form. **2**

(ii) Represent your solutions on an Argand diagram. **1**

(iii) Factorise z^6+1 as a sum of two cubes. (Don't try to factorise further) **1**

(iv) Hence, write down the roots of the equation $z^2 = z^4 + 1$ in the form $a+ib$. **2**

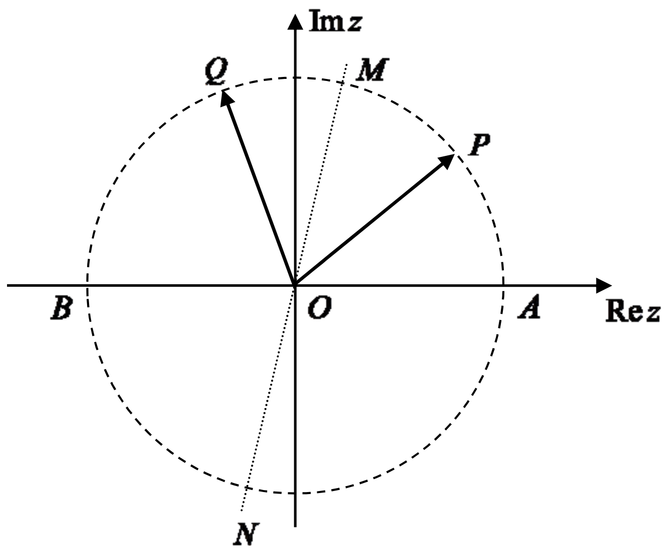
(b) (i) If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$. **1**

(ii) Hence, show that $\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$. **3**

Question 7 (continued)

(c) P and Q are two points on the unit circle representing complex numbers $z_1 = \text{cis}\alpha$ and $z_2 = \text{cis}\beta$ respectively.

The line MN is the bisector of $\angle POQ$.



(i) The point R represents the complex number $z_1 z_2$. Explain why R also lies on the unit circle. **1**

(ii) Hence explain using arguments of complex numbers why the complex number $1 + z_1 z_2$ must lie on MN. **2**

(iii) Assuming $z_1 z_2 \neq -1$, explain why $\dfrac{z_1 + z_2}{1 + z_1 z_2}$ is real. **2**

(d) (i) Sketch the locus of z given that $\arg\left(1 - \dfrac{1}{z}\right) = \dfrac{\pi}{3}$. **2**

(ii) Hence, show that points on this locus also satisfy $|z|^2 + |z-1|^2 = |z^2 - z| + 1$. **2**

End of Paper

Section I Multiple Choice

1. D

$\left|z-(-2+3i)\right|=4$

The distance of z from $(-2,3)$ is 4 units.

2. C

$$\begin{aligned}(1+i)^{12} &= \left(\sqrt{2}\text{cis}\tfrac{\pi}{4}\right)^{12} \\ &= 64\text{cis}\left(\tfrac{12\pi}{4}\right)=64\text{cis}\pi \\ &= -64\end{aligned}$$

3. C

$$\left||z_1|-|z_2|\right| \le |z_1+z_2| \le |z_1|+|z_2|$$
$$|3-7| \le |z_1+z_2| \le |10|$$

4. B

$\overrightarrow{PQ}=(a+bi)$

$\overrightarrow{PR}=i.\overrightarrow{PQ}=-b+ai$

$$\overrightarrow{OR}=\overrightarrow{OP}+\overrightarrow{PR}=(1+i)+(-b+ai)$$
$$=(1-b)+i(1+a)$$

5. A

Let the other root be α

Sum of roots: $-1+2i+\alpha=2i \Rightarrow \alpha=1$

Product of roots: $k=(-1+2i)\times 1=-1+2i$

Alternately, sub $x=-1+2i$ into the equation and solve for *k*.

Section II

Question 6

(a) (i)

$$z^2 = (7-2i)^2 = 49 - 28i - 4$$
$$= 45 - 28i$$

(ii) $2i\overline{z} = 2i(7+2i) = -4 + 14i$

Well done.

(b)

(i) Let $(x+iy)^2 = -5-12i$

$x^2 - y^2 + 2ixy = -5 - 12i$

Equating real and imaginary parts

$x^2 - y^2 = -5$ --- (1)
$2xy = -12$
$xy = -6$ --- (2)

By inspection, $x = 2, y = -3$ or $x = -2, y = 3$

Alternately, from (2) $y = \frac{-6}{x}$. Sub in (1)

$$x^2 - \frac{36}{x^2} = -5$$
$$x^4 + 5x^2 - 36 = 0$$
$$(x^2+9)(x^2-4) = 0$$
$$x = \pm 2 \text{ as } x \in \mathbb{R}$$

$$x = 2 \Rightarrow y = -3$$
$$x = -2 \Rightarrow y = 3$$

Therefore the square roots are $2-3i$ and $-2+3i$

ALTERNATELY

$$-5-12i = 2^2 + (3i)^2 - 2\times 2\times 3i$$
$$= (2-3i)^2$$

Therefore the square roots are $2-3i$ and $-2+3i$

(ii) $z^2 + 5iz - 5 + 3i = 0$

$$\Delta = 25i^2 - 4(1)(-5+3i)$$
$$= -5 - 12i$$

From (i) $(2-3i)^2 = (-2+3i)^2 = -5-12i$

$$z = \frac{-5i+(2-3i)}{2}, \frac{-5i+(-2+3i)}{2}$$
$$z = 1-4i, -1-i$$

Mostly well done. Some careless algebraic errors were costly here.

(c)

(i) $z = 1+i$ and $w = 1-i\sqrt{3}$

$$\frac{z}{w} = \frac{1+i}{1-i\sqrt{3}} \times \frac{1+i\sqrt{3}}{1+i\sqrt{3}}$$
$$= \frac{1+i\sqrt{3}+i-\sqrt{3}}{1+3}$$
$$= \frac{\left(1-\sqrt{3}\right)+i\left(1+\sqrt{3}\right)}{4}$$

(ii) $z = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ and $w = 2\text{cis}\left(-\frac{\pi}{3}\right)$

(iii) Using the result from (ii)

$$\frac{z}{w} = \frac{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)}{2\text{cis}\left(-\frac{\pi}{3}\right)}$$
$$= \frac{1}{\sqrt{2}}\text{cis}\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$$
$$= \frac{1}{\sqrt{2}}\text{cis}\left(\frac{7\pi}{12}\right)$$

c) Parts i. ii and iii. Well done.

(iv) From (ii) and (iii)

$$\frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)=\frac{1-\sqrt{3}}{4}+i\frac{1+\sqrt{3}}{4}$$

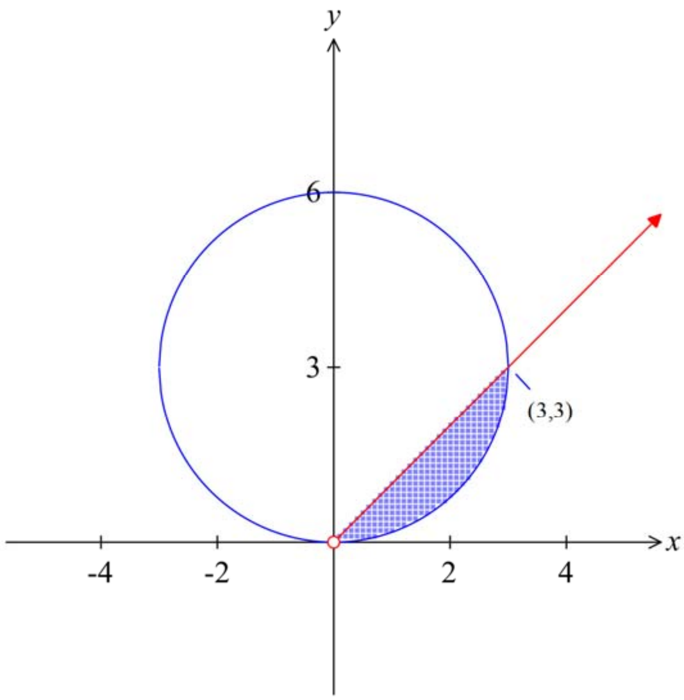
Equating imaginary parts,

$$\frac{1}{\sqrt{2}}\sin\frac{7\pi}{12}=\frac{1+\sqrt{3}}{4}$$

$$\sin\frac{7\pi}{12}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

iv. A number of students correctly equated imaginary parts but then made an error in the final line of working when they needed to multiply though by $\sqrt{2}$. Take care with this.

(d)



Students needed to show the centre of the circle at $\mathrm{Im}(z)=3$.

½ mark was deducted if the ray did not look like it was going to pass through (3,3).

(e)

$\overrightarrow{OR} \equiv z$ then, $\overrightarrow{OP} \equiv -2iz$

$\overrightarrow{OR} + \overrightarrow{OP} = \overrightarrow{OQ}$

$z - 2iz = -1 + 5i$

$z(1-2i) = -1 + 5i$

$$z = \frac{-1+5i}{1-2i} = \frac{(-1+5i)(1+2i)}{5}$$

$$= \frac{-1-10+3i}{5} = -\frac{11}{5} + \frac{3}{5}i$$

A number of students incorrectly stated that the diagonals of a rectangle bisect the vertex angles.
Clearly distinguish between when you are referring to sides of the rectangle and vectors. (Use the correct notation for a vector)
A lot of students correctly obtained the equation $z - 2iz = -1 + 5i$ in some form. Rather than factorising and making z the subject, they substituted $x + iy$ for z and then equated real and imaginary parts. While this does eventually lead to the correct solution it is time consuming and some students did make errors.

Question 7

(a) $z^6 = -1$

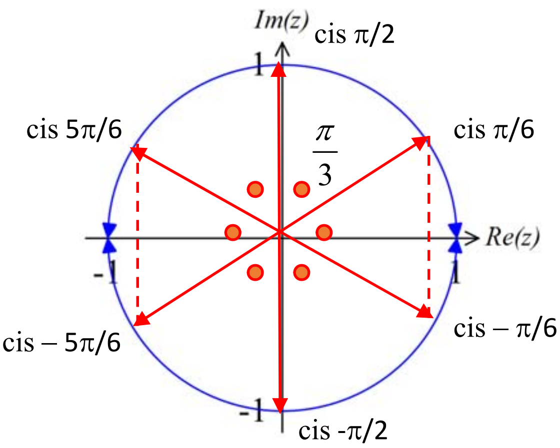
(i) $z^6 = \text{cis}(\pi + 2k\pi) = \text{cis}(2k+1)\pi, \quad k \in \mathbb{Z}$

$z = \text{cis}\dfrac{(2k+1)\pi}{6}$ using de Moivre's theorem

$z = \text{cis} \pm \dfrac{\pi}{6}, \text{cis} \pm \dfrac{\pi}{2}, \text{cis} \pm \dfrac{5\pi}{6}$

Generally well done. A small number of students started with the incorrect mod-arg form for – 1 and could not arrive at the correct roots. Although no penalty was incurred, many students are not referencing de Moivre's theorem and also do not state that *k* is an integer.

(ii) The six roots are equally spaced on the unit circle.



Generally well done. Some students drew no unit circle or drew a circle without indicating its size. This was penalised. There was no penalty for not indicating conjugate relationships or marking equal angles between roots although this is desirable.

When marking roots of unity on an Argand diagram – draw a unit circle, indicate all the roots clearly by labelling them as shown above, indicate that they are equally spaced and show any conjugate relationships using dotted lines as shown in the diagram above.

(iii) $z^6 + 1 = (z^2)^3 + 1^3 = (z^2+1)(z^4 - z^2 + 1)$

Surprisingly, many students failed to understand what is meant by factorise using sum of cubes. The reference is to $a^3 + b^3$.

(v) $z^6 = -1 \Rightarrow z^6 + 1 = 0$

From (iii) $(z^2+1)(z^4 - z^2 + 1) = 0$

$z^2 + 1 = 0$ or $z^4 - z^2 + 1 = 0$

Therefore, the roots of $z^4 - z^2 + 1 = 0$ are the same as the roots of $z^6 + 1 = 0$ except $\text{cis} \pm \frac{\pi}{2}$ which correspond to $z^2 + 1 = 0$

ie $z = \text{cis} \pm \frac{\pi}{6}, \text{cis} \pm \frac{5\pi}{6}$. Therefore the roots are

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad \frac{\sqrt{3}}{2} - \frac{1}{2}i, \quad -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

This was a "Hence, … write down…" question. The "write down" is telling you that you do not need to work out the roots from scratch. The "Hence" is telling you that there is a link to previous parts of the question. Superior responses clearly explained the link to the previous part and then listed the roots. If the link was not explained, full credit was not awarded.

Even where the roots were identified correctly with reasoning, some students did not note that the roots were required in Cartesian form and lost one mark.

Students who did not use the "Hence" could earn a maximum of one mark (as this was not a "hence or otherwise") provided they arrived at all four roots correctly.

(b)

(i)

$$\begin{aligned} z^n + z^{-n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) && \text{using deMoivre's theorem} \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta && \because \cos(x) \text{ is even; } \sin(x) \text{ is odd} \\ &= 2\cos n\theta \end{aligned}$$

Generally well done. Again, many students do not communicate clearly how they get from one line of working to the next. Good communication makes it easy for the marker to follow your working and you are less likely to lose marks because the marker cannot follow how you got the next line of working.

(ii)

$$\begin{aligned} (z + z^{-1})^5 &= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5} \\ (2\cos\theta)^5 &= (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1}) \\ 32\cos^5\theta &= 2\cos 5\theta + 5(2\cos 3\theta) + 10(\cos\theta) && \text{using result from (i)} \\ \cos^5\theta &= \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta) \end{aligned}$$

Poorly done. Many students did not know how to use part (i) and started with $(\cos\theta + i\sin\theta)^5$. In the absence of other ideas, this was not a bad place to start and it is possible to arrive at the required result although the solution is very long and tedious. Students using this approach generally only made partial progress with very few students managing to get the required result. Full marks were awarded to the successful few, despite this being a "Hence" and not a "Hence or otherwise".

(c) (i) $|z_1 z_2| = |z_1| \cdot |z_2|$

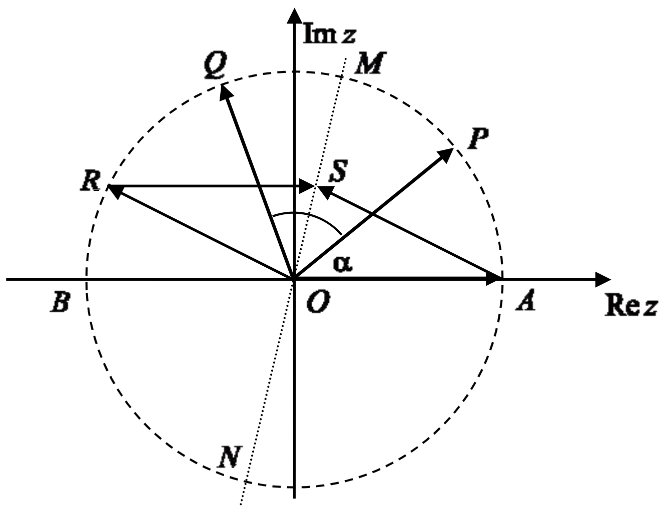
$= 1 \times 1$

$= 1$

So R is one unit from the origin, ie. it lies on the unit circle

Most students knew why *R* lies on the unit circle although once again communicating using the notation and language appropriately was a challenge for many. For instance, $\|R\|$ does not make sense, as *R* is the name of a point and not a complex number. Use $\|z_1 z_2\|$ or $\|\overrightarrow{OR}\|$. However, this was marked very leniently with most students achieving the one mark.

(ii)



$\arg z_1 = \alpha$ and $\arg z_2 = \beta$ ie $\angle AOP = \alpha$ and $\angle AOQ = \beta$.

So $\angle POQ = \beta - \alpha$.

Now, *R* is the point representing $z_1 z_2$.

$\angle AOR = \arg z_1 z_2$

$= \arg z_1 + \arg z_2$

$= \alpha + \beta$

Let $\overrightarrow{OS}$ be the vector representing the sum of $\overrightarrow{OR}$ and $\overrightarrow{OA}$ ie *S* corresponds to $1 + z_1 z_2$.

$OASR$ is a rhombus (opposite sides parallel, adjacent sides $OA = OR = 1$)

$\therefore \quad \angle AOS = \frac{1}{2}\angle AOR$ (diagonals of rhombus bisect vertex angles)

$= \frac{\alpha + \beta}{2}$

$\angle MOA = \angle MOP + \angle POA$

$= \frac{1}{2}\angle MOP + \alpha$ as MP bisects $\angle POQ$

$\therefore$ $= \frac{\beta - \alpha}{2} + \alpha$

$= \frac{\alpha + \beta}{2}$

This is the angle made by line *MN*. $\therefore S$ lies on *MN* or $1 + z_1 z_2$ lies on *MN*.

> Quality of responses was variable. Communication and proper use of notation needs more work. $\arg(\alpha)$ makes no sense. α is an argument already. $\arg(1 + z_1 z_2) \neq \arg 1 + \arg(z_1 z_2)$. It is equal to $\frac{\arg(z_1) + \arg(z_2)}{2}$ but this is not obvious and must be justified. Diagonals of a parallelogram do not bisect the vertex angles. Diagonals of a rhombus do. Establish why you have a rhombus.
>
> There are points that are named in the question. Do not use a different letter for $z_1 z_2$ than the one given in the question. If you are constructing or adding other points, <u>draw a diagram</u>.
>
> To earn full credit, you needed to:
>
> - Establish that $\angle MOA = \frac{\alpha + \beta}{2}$ using appropriate reasoning by using the given fact that *MN* bisects $\angle POQ$
> - Establish that the argument of $1 + z_1 z_2$ is also $\frac{\alpha + \beta}{2}$ again using appropriate reasoning.
> - Hence, conclude that $1 + z_1 z_2$ lies on *MN*.
>
> Simply stating the above results was inadequate.

(iii) $OP + OQ$ also lies on MN (or *MN* produced)

Construct parallelogram *OPTQ*. $\overrightarrow{OT}$ represents $\overrightarrow{OP} + \overrightarrow{OQ}$ or $z_1 + z_2$.

OPTQ is a rhombus (as *OP* = *OQ* = 1 radii of unit circle)

Therefore, *OT* bisects $\angle POQ$ (diagional of a rhombus bisect vertex angles) or *T* lies on *MN*.

So, both $T \equiv z_1 + z_2$ and $S \equiv 1 + z_1 z_2$ lie on *MN*.

ie

$$\arg(1 + z_1 z_2) = \arg(z_1 + z_2) \pm \pi$$

$$\arg(z_1 + z_2) - \arg(1 + z_1 z_2) = 0 \text{ or } \pi$$

$$\arg\left(\frac{z_1 + z_2}{1 + z_1 z_2}\right) = 0 \text{ or } \pi$$

ie. $\frac{z_1 + z_2}{1 + z_1 z_2}$ is real.

Alternately, as both both $T \equiv z_1 + z_2$ and $S \equiv 1 + z_1 z_2$ have the same argument, say θ, then:

$1 + z_1 z_2 = k cis\theta$ and $z_1 + z_2 = l cis\theta$ and $\dfrac{z_1 + z_2}{1 + z_1 z_2} = \dfrac{l}{k}$ which is real.

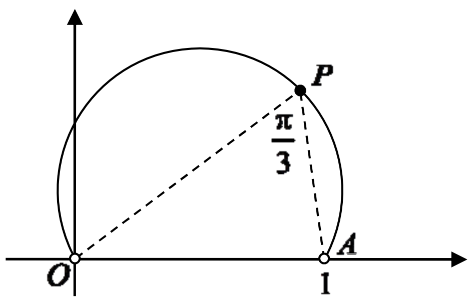
> Many students incorrectly state that $\arg(z_1 + z_2) = \arg(z_1) + \arg(z_2)$. Some students started with
>
> $\dfrac{z_1 + z_2}{1 + z_1 z_2} = k$ where k is real. This is what you had to prove. You cannot assume it.
>
> To earn full credit, you had to
>
> - Establish that the argument of $z_1 + z_2$ is also $\dfrac{\alpha + \beta}{2}$ using appropriate reasoning.
> - Conclude that both $1 + z_1 z_2$ and $z_1 + z_2$ lie on *MN*.
> - Hence, conclude that the difference in their arguments is 0 (see note).
> - Hence, conclude that $\arg\left(\dfrac{z_1 + z_2}{1 + z_1 z_2}\right) = 0$ and therefore that $\dfrac{z_1 + z_2}{1 + z_1 z_2}$ must be real.
>
> Note: The difference in arguments can actually be 0 or π depending on the location of *P* and *Q*. However, there was no penalty for not listing π.

(d) (i)

$$\arg\left(1 - \frac{1}{z}\right) = \frac{\pi}{3}$$

$$\arg\frac{z-1}{z} = \frac{\pi}{3}$$

$$\arg(z-1) - \arg z = \frac{\pi}{3}$$



> Generally, well done. A small number of students drew the wrong arc or a minor arc. Penalties applied for not drawing open circles at the end-points of the arc, for drawing an arc that did not cross the *y*-axis, or for not indicating the size of the angle in the segment.
>
> Deriving the centre and radius was not necessary and some students wasted precious time on finding these.
>
> Students who proceeded to derive the locus algebraically either were not successful or arrived at the entire circle.

(ii) Using the Cosine Rule in ΔOAP :

$$\begin{aligned} OA^2 &= OP^2 + AP^2 - 2OP \cdot AP \cdot \cos\frac{\pi}{3} \\ 1 &= |z|^2 + |z-1|^2 - 2|z| \cdot |z-1| \cdot \frac{1}{2} \\ 1 &= |z|^2 + |z-1|^2 - |z(z-1)| \\ |z|^2 + |z-1|^2 &= |z^2 - z| + 1 \end{aligned}$$

Poorly done. Students who saw the relationship in the triangle, were able to derive the result with no difficulty. Algebraic approaches were not successful and could not be awarded any credit.

The "Hence" in the question was a clue that you had to use the diagram from part (i). If you see a z and $z-1$ in the result, try to locate where these are on your diagram. This will help you find any geometrical relationships.

End of Solutions